

Rate of Convergence for Large Coupling Limits in Sobolev Spaces

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1 Introduction and Overview

The main objective of this article is to obtain estimates of the convergence rates for certain singularly perturbed parabolic boundary value problems. More concretely, let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, be a bounded open subset with smooth boundary Γ . Denote by A the self adjoint realization of the Laplacian, Δ , in $L^2(\Omega)$ with Neumann boundary conditions on Γ . As is well known, the operator A generates a positive semigroup which we denote formally by e^{-tA} . This semigroup corresponds to a reflected diffusion process with reflection at Γ .

Now, let $\Omega_0 \subset\subset \Omega$ be a compact inclusion with boundary Γ_0 . Write $\Omega_1 := \Omega \setminus \overline{\Omega}_0$, so that $\Omega = \Omega_0 \cup \Gamma_0 \cup \Omega_1$ (see Figure 1). We consider Schrödinger type operators of the form: $A_\lambda := A - \lambda \mathbf{1}_{\overline{\Omega}_0}$; where λ is a positive parameter and $\mathbf{1}_E(x)$ is the characteristic function of the measurable set E . Such operators also determine (parameterized) semigroups e^{-tA_λ} and our goal is to characterize the following limit: $\lim_{\lambda \rightarrow \infty} e^{-tA_\lambda}$.

The semigroups e^{-tA_λ} corresponds to a reflected diffusion process, with reflection at Γ which could also get “killed” or absorbed, on entering the region Ω_0 . We expect that as $\lambda \rightarrow \infty$ this absorption occurs quicker, on average, so that, at least formally, $\lambda = \infty$ corresponds to *instantaneous absorption*. In other words, if we denote by B the realization of the Laplacian in $L^2(\Omega_1)$ with Neumann boundary conditions at Γ (reflection) and Dirichlet boundary conditions at Γ_0 (instantaneous absorption), we should then have: $\lim_{\lambda \rightarrow \infty} e^{-tA_\lambda} = e^{-tB}$.

In addition to understanding the large λ limit, one would also like to quantify in what manner, i.e. norm, and at what rate one gets convergence. In general, the choice of norm will affect the rates one obtains. The problem is complicated by the fact that the semigroups are defined on different domains: e^{-tA_λ} on $L^2(\Omega)$ while e^{-tB} is defined on $L^2(\Omega_1)$.

Large coupling limits have been studied previously in the literature and the analysis seems to split roughly into:

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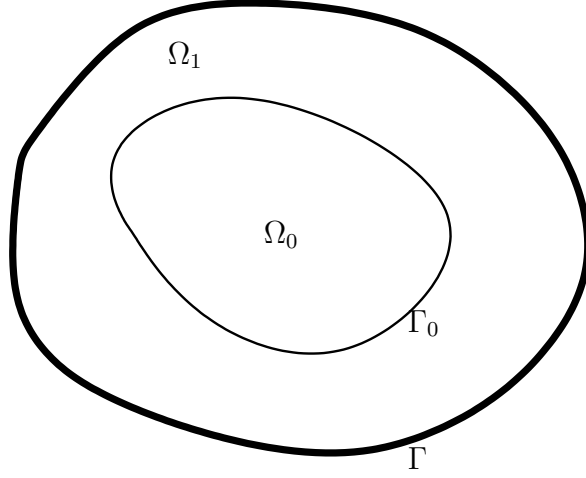


Figure 1: The region Ω

- i) the assumptions made on the domains Ω_0 and Ω , their boundaries, dimensions, e.t.c.;
- ii) whether one studies the convergence of the operators $A_\lambda \rightarrow B$, their resolvents $(A_\lambda - zI)^{-1} \rightarrow (B - zI)^{-1}$, or the closely related semigroups $e^{-tA_\lambda} \rightarrow e^{-tB}$; and,
- iii) the allowable class of “interaction potentials”.

The works of DEMUTH ET AL [4, 5, 6, 7] contain the most complete results that we are aware of. It is important to note that they study the case $\Omega = \mathbb{R}^m$ which is quite different from the case considered here. Their main technical tool is the Feynman-Kac formula and the use of *stochastic spectral analysis* - in particular estimates for occupation and hitting times of Brownian motion.

This problem arose for us in the context of *stochastic reaction diffusions* and comparing different mechanisms for capturing biochemical reactions. Indeed, the case described above corresponds to a diffusing particle searching to undergo a bimolecular annihilation reaction, $X + Y \rightarrow \emptyset$, by getting absorbed by the stationary “target”, Ω_0 . The reaction mechanism can be given, on the one hand, by the interaction potential, $\lambda \mathbf{1}_{\overline{\Omega_0}}$, and, on the other hand, by the Dirichlet boundary condition on Γ_0 .

Our own approach is to study the associated parabolic problems:

$$\frac{\partial \rho}{\partial t} = \kappa \Delta \rho(t, x), \quad x \in \Omega_1, \quad t > 0, \quad (1)$$

with the interior Dirichlet boundary condition

$$\rho(t, x) = 0, \quad x \in \Gamma_0, \quad t > 0, \quad (2)$$

and the exterior boundary condition

$$\nabla \rho(t, x) \cdot \hat{\mathbf{n}} = 0, \quad x \in \Gamma, \quad t > 0, \quad (3)$$

and initial condition $\rho(0, x) = g(x)$. We interpret $\rho(t, x)$ as the probability density that the *unreacted* diffusing particle, of diffusivity $\kappa > 0$, is located at $x \in \Omega_1$ at time t . $g(x)$ is the initial distribution of the reactant and from our previous discussion we note that $\rho(t, x) = e^{-\kappa t B}(g(x))$.

We will also consider:

$$\frac{\partial p_\lambda}{\partial t} = \kappa \Delta p_\lambda(t, x) - \lambda \mathbf{1}_{\overline{\Omega}_0}(x) p_\lambda(t, x), \quad x \in \Omega, t > 0, \quad (4)$$

now with only the exterior boundary condition (3), and the modified initial condition

$$p_\lambda(0, x) = E_0[g](x) := \begin{cases} g(x), & x \in \Omega_1; \\ 0, & x \in \Omega_0. \end{cases}$$

Let $R_i f := f|_{\Omega_i}$, $i = 0, 1$, be the restriction operators which we need in order to compare both solutions. One of the main results of this paper is the following estimate:

$$\|(R_1 \circ e^{-\kappa t A_\lambda} \circ E_0)[g] - e^{-\kappa t B}[g]\| := \|p_\lambda|_{\Omega_1} - \rho\| = \mathcal{O}(\lambda^{-\frac{1}{4}}), \quad (5)$$

where all the norms are taken in $L^2([0, T] \times \Omega_1)$, with $0 < T < \infty$ is arbitrary but fixed.

We now summarize the contents of this paper:

In Section 2, we collect some facts concerning our main tools - Sobolev spaces, Interpolation and Trace theorems as well as a potent characterization of the extension operator, E_0 and the restriction operators, R_i . This section is rather long and no new results are proved, but it contains various facts we will need in our analysis.

In Section 3, we define the notion of a generalized solution and we give an equivalent and very useful formulation of the problem (4) as a system of boundary coupled equations and derive some important a priori estimates. These estimates lay the foundation for the subsequent results and are proved by the fairly standard “energy method”.

In Section 4, we study the weak convergence of p_λ using the apriori estimates. The results here are straightforward application of the general facts from Section 2.

In Section 5, we study the regularity of a related “transposed” problem in the exterior domain, Ω_{ext} and use it to strengthen the convergence of p_λ in the Sobolev spaces considered.

Roughly speaking, the overall program is that we first derive estimates in the interior Ω_0 . We then transfer these estimates to the boundary, Γ_0 . We then use this estimate on the boundary to derive estimates in the exterior region.

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2 Preliminaries

We pause here to review some facts we will need later. Much of this material is fairly standard and we go through it mostly to fix notation. We essentially follow the treatment given in LIONS & MAGENES [10], ADAMS [1] and McLEAN [11].

2.1 Sobolev Spaces

We start with a brief description of the Sobolev-Slobodeckii spaces. Let V be an arbitrary open subset of \mathbb{R}^m . As usual, $L^2(V)$ will denote the space of square integrable measurable functions, u , on V . For $k \in \mathbb{N}$, $H^k(V)$ will denote the Sobolev space defined as:

$$H^k(V) = \{u \mid D^\alpha u \in L^2(V) \forall \alpha \text{ with } |\alpha| \leq k\}$$

where, $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi index with $|\alpha| = \alpha_1 + \dots + \alpha_m$, and $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}$ denotes a mixed (weak) distributional derivative of order $|\alpha|$. We endow $H^k(V)$ with the norm:

$$\|u\|_{H^k(V)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2(V)}^2 \right)^{\frac{1}{2}}.$$

It is also standard to define $H^0(V) := L^2(V)$. We now give an “intrinsic” definition of fractional order Sobolev-Slobodeckii spaces due to SLOBODECKII [14]. Let V be as before and first assume that $0 < \mu < 1$. We define the Slobodeckii semi-norm:

$$[u]_{\mu,2,V} := \left(\int_V \int_V \frac{|u(x) - u(y)|^2}{|x - y|^{m+2\mu}} dx dy \right)^{\frac{1}{2}}.$$

Now if $r = k + \mu$ where k is nonnegative integer and μ is its positive fractional part, we define

$$H^r(V) = \{u \mid [D^\alpha u]_{\mu,2,V} < \infty, \forall \alpha \text{ with } |\alpha| = k\},$$

with the norm

$$\|u\|_{H^r(V)} := \left(\|u\|_{H^k(V)}^2 + \sum_{|\alpha|=k} [D^\alpha u]_{\mu,2,V}^2 \right)^{\frac{1}{2}}.$$

In addition, we shall also deal with the Banach-space valued function spaces. Let $0 < T < \infty$ be a real number and $I := (0, T)$ be the associated open interval. If X is an arbitrary Banach space with norm $\|\cdot\|_X$, $L^2(I; X)$ consists of functions $u(\cdot, t)$ that take values in X for almost every $t \in I$ and such that the L^2 norm of $\|u(t, \cdot)\|_X$ is finite. We then define the norm as follows:

$$\|u\|_{L^2(I; X)} := \left(\int_0^T \|u(t, \cdot)\|_X^2 dt \right)^{1/2} \quad (6)$$

Just as before, if l is a positive integer we can define

$$H^l(I; X) = \{u \mid \frac{d^j u}{dt^j} \in L^2(I; X) \text{ for } 0 \leq j \leq l\}$$

with the norm

$$\|u\|_{H^l(I; X)} = \left(\sum_{j=0}^l \left\| \frac{d^j u}{dt^j} \right\|_{L^2(I; X)}^2 \right)^{\frac{1}{2}}.$$

We also define the fractional order Sobolev spaces for Banach-space valued functions similar to the usual case. First the Slobodeckii semi norm is defined as

$$[u]_{\mu, 2, X} := \left(\int_0^T \int_0^T \frac{\|u(t, \cdot) - u(\tau, \cdot)\|_X^2}{|t - \tau|^{1+2\mu}} dt ds \right)^{\frac{1}{2}},$$

and for $s = l + \mu$, we define the norm

$$\|u\|_{H^s(I; X)} := \left(\|u\|_{H^l(I; X)}^2 + [D_t^l u]_{\mu, 2, X}^2 \right)^{\frac{1}{2}}.$$

Define the cylinder $V_T = I \times V$. Let $r, s \geq 0$. In what follows we will be most interested in the spaces $H^{r,s}(V_T) := L^2(I; H^r(V)) \cap H^s(I; L^2(V))$ with the norm

$$\|u\|_{H^{r,s}(V_T)} := \left(\int_0^T \|u(t, \cdot)\|_{H^r(V)}^2 dt + \int_V \|u(\cdot, x)\|_{H^s(I)}^2 dx \right)^{\frac{1}{2}}.$$

2.2 Trace Theorems and Sobolev Spaces on the Boundary

Since we will be studying boundary problems, we will now give a way to define Sobolev spaces on the boundary of domains. We begin with the case of Euclidean space. Following [10] for $r, s \geq 0$ let

$$H^{r,s}(\mathbb{R}^{m+1}) := \mathbb{R}_t \times \mathbb{R}_x^m := L^2(\mathbb{R}; H^r(\mathbb{R}^m)) \cap H^s(\mathbb{R}; L^2(\mathbb{R}^m))$$

be the space of tempered distributions on \mathbb{R}^{m+1} such that if we let $\widehat{u}(\tau, \xi)$ denote the Fourier transform of u in (t, x) respectively, we have that

$$\|u\|_{H^{r,s}(\mathbb{R}^{m+1})}^2 := \int_{\mathbb{R}^{m+1}} \left[(1 + \tau^2)^s + (1 + |\xi|^2)^r \right] |\widehat{u}(\tau, \xi)|^2 d\tau d\xi < \infty.$$

The above norm turns out to be equivalent to the one previously defined “intrinsically”.

As usual $\mathcal{D}(\mathbb{R}^m)$ is the space of smooth functions in \mathbb{R}^m with compact support. If $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ we write $x' = (x_1, \dots, x_{m-1})$. The following theorem is well known, see for example [10].

Theorem 2.1 (Parabolic Trace Theorem). *For*

1. $r > \frac{1}{2}$, $s \geq 0$, the restriction map

$$\begin{aligned} \mathcal{D}(\mathbb{R}^{m+1}) &\rightarrow \mathcal{D}(\mathbb{R}^m) \\ \phi(t, x', x_m) &\mapsto \left. \frac{\partial^j \phi}{\partial x_m^j} \right|_{(t, x', 0)} \end{aligned}$$

extends by density to a bounded linear map

$$\mathcal{T}_j^x : H^{r,s}(\mathbb{R} \times \mathbb{R}^m) \rightarrow H^{\mu_j, \nu_j}(\mathbb{R} \times \mathbb{R}^{m-1})$$

$$\text{where } \mu_j = r - j - \frac{1}{2} \text{ and } \nu_j = s \left(1 - \frac{j}{r} - \frac{1}{2r} \right)$$

2. and $s > \frac{1}{2}$, $r \geq 0$, the restriction map

$$\begin{aligned} \mathcal{D}(\mathbb{R}^{m+1}) &\rightarrow \mathcal{D}(\mathbb{R}^m) \\ \phi(t, x) &\mapsto \phi(0, x) \end{aligned}$$

also extends to a bounded linear map

$$\mathcal{T}_0^t : H^{r,s}(\mathbb{R} \times \mathbb{R}^m) \rightarrow H^{r(1-\frac{1}{2s})}(\mathbb{R}^m)$$

Note that the traces are taken in the sense of distributions and not in any point wise sense.

Recall that a map is $C^{k,\gamma}$ for some integer $0 \leq k \leq \infty$ and $0 < \gamma \leq 1$ if it is k times continuously differentiable with Hölder continuous k -th order derivatives with exponent γ . Let $\mathbb{R}_+^m = \{x \in \mathbb{R}^m \mid x_m > 0\}$ and we identify the boundary $\partial \mathbb{R}_+^m := \{x \in \mathbb{R}^m \mid x_m = 0\}$ with \mathbb{R}^{m-1} . We then say that the domain $V \subset \mathbb{R}^m$ is of class $C^{k,\gamma}$ when every boundary point $x \in \omega := \partial V$ has an open neighborhood U with an associated $C^{k,\gamma}$ -diffeomorphism Φ (a bijective mapping with $C^{k,\gamma}$ inverse) such that $\Phi(x) = 0$, $\Phi(U \cap V) \subset \mathbb{R}_+^m$ and $\Phi(U \cap \omega) \subset \mathbb{R}^{m-1}$. In particular this implies that V is “locally on one side” of ω . Note that a $C^{0,1}$ domain is just a Lipschitz domain. Using such local charts and a partition of unity we then obtain the following corollary to Theorem 2.1:

Corollary 2.1. *Let ω be a $C^{k-1,1}$ boundary and $\frac{1}{2} < r \leq k$, $s \geq 0$, then there exists a bounded trace map:*

$$\begin{aligned} \mathcal{T}_0^x : H^{r,s}(V_T) &\rightarrow H^{r-\frac{1}{2}, s(1-\frac{1}{2r})}(\omega_T) \\ u &\mapsto u|_{\omega_T} \end{aligned}$$

where $\omega_T = I \times \omega$ is the lateral boundary.

We will also make use of the following well known estimate:

Theorem 2.2 (Modified Trace Theorem). *Let V be bounded open subset of \mathbb{R}^m with $C^{0,1}$ boundary, ω . Then there exist a bounded linear operator $\mathcal{T}_0^x : H^{1,0}(V_T) \rightarrow L^2(\omega_T)$ such that given $\epsilon > 0$ and arbitrary $u \in H^{1,0}(V_T)$ there exists a constant C_ϵ (independent of u) with*

$$\|\mathcal{T}_0^x[u]\|_{L^2(\omega_T)} \leq C_\epsilon \|u\|_{L^2(V_T)} + \epsilon \|Du\|_{L^2(V_T)}. \quad (7)$$

2.3 Real Interpolation Spaces

Let X_0, X_1 , with X_1 dense in X_0 , be normed spaces contained in a vector space \mathcal{V} . Equip the spaces $X_0 \cap X_1$ and $X_0 + X_1$ with the norms:

$$\begin{aligned} \|u\|_{X_0 \cap X_1}^2 &:= \|u\|_{X_0}^2 + \|u\|_{X_1}^2 \\ \|u\|_{X_0 + X_1}^2 &:= \inf_{\substack{u=u_0+u_1 \\ u_0 \in X_0; u_1 \in X_1}} (\|u_0\|_{X_0}^2 + \|u_1\|_{X_1}^2) \end{aligned}$$

Consider the K functional for $t > 0$ defined by:

$$K(t, u) := \inf_{\substack{u=u_0+u_1 \\ u_0 \in X_0; u_1 \in X_1}} (\|u_0\|_{X_0}^2 + t^2 \|u_1\|_{X_1}^2)^{\frac{1}{2}},$$

and, for $0 < \theta < 1$, $1 \leq q < \infty$ the (real) interpolation spaces $X_{\theta,q} := [X_0, X_1]_{\theta,q}$ is the set of functions with norm:

$$\|u\|_{X_{\theta,q}} := \left(\int_0^\infty |t^{-\theta} K(t, u)|^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

(Note that some authors define the spaces $\tilde{X}_{\theta,q} = [X_1, X_0]_{\theta,q}$ but it is a simple matter to show that $\tilde{X}_{\theta,q} = X_{1-\theta,q}$). The main results we will need are:

Lemma 2.1. *Let $u \in X_0 \cap X_1$, then $u \in X_{\theta,q}$ and there exists a constant $C = C(\theta, q)$ such that*

$$\|u\|_{X_{\theta,q}} \leq C \|u\|_{X_0}^{1-\theta} \|u\|_{X_1}^\theta$$

and

Lemma 2.2. *For $0 < \theta < 1$, the fractional order Sobolev-Slobodeckii space can be characterized by interpolation as*

$$H^\theta(\mathbb{R}^m) = [L^2(\mathbb{R}^m), H^1(\mathbb{R}^m)]_{\theta,2}$$

with equivalent norms

The interpolation theorem gives another characterization of the Sobolev spaces and gives us leeway in dealing with the norms.

2.4 The Extension and Restriction Maps

Recall the maps (same notation as the Introduction)

$$\begin{aligned} R_1 : L^2(\Omega) &\rightarrow L^2(\Omega_1) \\ f &\mapsto f|_{\Omega_1} \end{aligned}$$

and

$$\begin{aligned} E_0 : L^2(\Omega_1) &\rightarrow L^2(\Omega) \\ g &\mapsto \begin{cases} 0, & x \in \Omega_0 \\ g(x), & x \in \Omega_1 \end{cases} \end{aligned}$$

It is easily checked that both E_0 and R_1 are bounded linear operators. Note that $R_1 E_0 [g] = g$ for all $g \in L^2(\Omega_1)$. However, one can check that $E_0 [R_1 f] = f$ if and only if $f = 0$ for a.e $x \in \Omega_0$. Since $H^1(\Omega_0) \subset L^2(\Omega_0)$ and $H^1(\Omega) \subset L^2(\Omega)$ both maps act naturally on the subspaces $H^1(\Omega_0)$ and $H^1(\Omega)$. But, as we shall soon see, while R_1 maps $H^1(\Omega)$ into $H^1(\Omega_1)$, E_0 maps $H^1(\Omega_0)$ into a larger set. Indeed, we have the following lemma which is a multidimensional analogue of the fact the derivative of the Heaviside function $\mathbf{1}_{(0,\infty)}(x)$ ($x \in \mathbb{R}$) is the Dirac delta distribution.

Proposition 2.1. *Let D denote the distributional derivative. Suppose that $f \in H^1(\Omega_1)$, then, as distributions, we have that*

$$D(E_0 [f]) = E_0 [Df] - \mathcal{T}_0^x[f] \hat{\mathbf{n}}(y) \otimes \delta(y \in \Gamma_0)$$

where $\hat{\mathbf{n}}$ is the unit outward pointing normal to the surface Γ_0 .

This allows us to deduce the following:

Corollary 2.2. $DE_0 [f] = E_0 [Df]$ iff $\mathcal{T}_0^x[f] = 0$.

What these two previous results basically say is that although we can decompose $L^2(\Omega)$ as the direct sum: $L^2(\Omega) = L^2(\Omega_0) \oplus L^2(\Omega_1)$ by writing $u = \{R_0 u, R_1 u\}$, in general $H^1(\Omega) \neq H^1(\Omega_0) \oplus H^1(\Omega_1)$.

3 Weak Solutions and *A priori* Estimates

3.1 Weak Solutions

We define $I = (0, T)$ and we also write

$$Q := I \times \Omega; \quad Q_0 := I \times \Omega_0; \quad Q_1 := I \times \Omega_1; \quad \Sigma_0 = I \times \Gamma_0; \quad \Sigma = I \times \Gamma.$$

Γ_0 will always be assumed to be at least a Lipschitz boundary and, for simplicity, Γ will be assumed smooth. We also use the slightly cumbersome notation for the time sliced domains:

$$\Omega_{\{t\}} := \{t\} \times \Omega; \quad \Omega_{0,\{t\}} := \{t\} \times \Omega_0; \quad \Omega_{1,\{t\}} := \{t\} \times \Omega_1.$$

For easier reference we now rewrite the equations as:

$$\frac{\partial \rho}{\partial t} = \kappa \Delta \rho(t, x), \quad (t, x) \in Q_1 \quad (8)$$

with initial and boundary conditions

$$\left. \begin{aligned} \rho(t, x) &= 0, & (t, x) &\in \Sigma_0 \\ \nabla \rho(t, x) \cdot \mathbf{n} &= 0, & (t, x) &\in \Sigma \\ \rho &= g, & \text{on } \Omega_{1, \{0\}} \end{aligned} \right\} \quad (9)$$

and

$$\frac{\partial p_\lambda}{\partial t} = \kappa \Delta p_\lambda(t, x) - \lambda \mathbf{1}_{\overline{\Omega}_0}(x) p_\lambda(t, x), \quad (t, x) \in Q \quad (10)$$

with initial and boundary conditions

$$\left. \begin{aligned} \nabla p_\lambda(t, x) \cdot \mathbf{n} &= 0, & (t, x) &\in \Sigma \\ p_\lambda &= E_0[g], & \text{on } \Omega_{\{0\}} \end{aligned} \right\} \quad (11)$$

Let us put, for convenience, $H_o^1(\Omega_1) = \{u \in H^1(\Omega_1) : u|_{\Gamma_0} = 0\}$ and recall the well known fact that for $u \in H_o^1(\Omega_1)$ we have the Poincaré type inequality

$$\|u\|_{L^2(\Omega_1)} \leq C \|\nabla u\|_{L^2(\Omega_1)},$$

so that the norms $\|u\|_{H_o^1(\Omega_1)} \asymp \|\nabla u\|_{L^2(\Omega_1)}$ are equivalent.

On a Lipschitz domain the classical Green's formulas hold. This is the basis for the following weak formulation of the initial-boundary value problem:

Definition 3.1. We say that ρ is a generalized solution of (8) and (9) if

(a) $\rho \in H_o^{1,0}(Q_1)$

(b)

$$\int_{Q_1} (\kappa \nabla \rho \cdot \nabla \psi - \rho \partial_t \psi) - \int_{\Omega_{1, \{0\}}} g \psi = 0$$

for all $\psi \in H^{1,1}(Q_1) \equiv H^1(Q_1)$ with $\psi|_\Sigma = 0$ and $\psi|_{\Omega_{1, \{T\}}} = 0$.

Here, (a) expresses the fact that the solution satisfies the Dirichlet boundary condition on Σ_0 while (b) says that ρ satisfies (8) in the weak sense and assumes the value g for $t = 0$.

Using the Lion's Projection Lemma (for a statement see COSTABEL [3, Lemma 2.1]), one can show the existence and uniqueness of the weak solution ρ as defined above. The argument is very similar to that in [3, Lemma 2.3].

Definition 3.2. We say that p_λ is a generalized solution of (10) and (11) if

(a) $p_\lambda \in H^{1,0}(Q)$

(b)

$$\int_Q (\kappa \nabla p_\lambda \cdot \nabla \phi - p_\lambda \partial_t \phi) - \int_{\Omega_{\{0\}}} E_0[g] \phi + \lambda \int_{Q_0} p_\lambda \phi = 0$$

for all $\phi \in H^1(Q)$ with $\phi|_{\Omega_{\{T\}}} = 0$ and where E_0 is the extension by 0 into Ω_0 .

A crucial fact is that we can also think of the Doi problem as a boundary coupled PDE as follows (see OLENIK [13], GIRSANOV [9] and LADYZHENSKAYA ET AL [12]):

Lemma 3.1. *Let $p_\lambda^+ = p_\lambda|_{\Omega_0} := R_0 p_\lambda$ and $p_\lambda^- = p_\lambda|_{\Omega_1} := R_1 p_\lambda$. Then the solution to the Doi problem can be obtained by finding $p_\lambda^+ \in H^{1,0}(Q_0)$ and $p_\lambda^- \in H^{1,0}(Q_1)$ such that*

$$\left. \begin{aligned} \frac{\partial p_\lambda^+}{\partial t} &= \kappa \Delta p_\lambda^+(t, x) - \lambda p_\lambda^+(t, x), & (t, x) \in Q_0, \\ \frac{\partial p_\lambda^-}{\partial t} &= \kappa \Delta p_\lambda^-(t, x), & (t, x) \in Q_1, \end{aligned} \right\} \quad (12)$$

with the coupling condition,

$$\left. \begin{aligned} p_\lambda^+(t, x) &= p_\lambda^-(t, x), & (t, x) \in \Sigma_0, \\ \nabla p_\lambda^+(t, x) \cdot \hat{\mathbf{n}} &= \nabla p_\lambda^-(t, x) \cdot \hat{\mathbf{n}}, & (t, x) \in \Sigma_0, \end{aligned} \right\} \quad (13)$$

and the external boundary condition $\nabla p_\lambda^-(t, x) \cdot \hat{\mathbf{n}} = 0$, for $x \in \Sigma$ as well as the initial conditions $p_\lambda^+(0, x) = 0$ and $p_\lambda^-(0, x) = g(x)$.

Proof. By Corollary 2.2, $p_\lambda \in H^{1,0}(Q)$ if and only if the first coupling condition in (13) holds. An integration by parts and the use of the second coupling condition gives the result. \square

3.2 A priori Estimates

We will now derive some simple integral estimates that the solutions to the model problem should satisfy. These estimates are at the heart of all the results in the subsequent sections.

Lemma 3.2 (Uniform L^2 Bounds). *Let $p_\lambda(t, x)$ satisfy (10). Given initial condition $E_0[g](x)$ in $L^2(\Omega)$ then p_λ is uniformly bounded in $L^\infty(I; L^2(\Omega))$ and $\nabla p_\lambda(t, x)$ is uniformly bounded in $L^2(Q)$. Moreover, $\|p_\lambda\|_{L^2(Q_0)} \rightarrow 0$ as $\lambda \rightarrow \infty$.*

Proof. Multiply (10) by $p_\lambda(t, x)$ and integrating over Ω we obtain

$$\int_{\Omega} \frac{\partial p_\lambda}{\partial t} p_\lambda \, dx = \kappa \int_{\Omega} p_\lambda \Delta p_\lambda \, dx - \lambda \int_{\Omega} \mathbf{1}_{\overline{\Omega}_0}(x) p_\lambda^2 \, dx$$

Hence it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} p_\lambda^2 \, dx = \kappa \int_{\Omega} p_\lambda \Delta p_\lambda \, dx - \lambda \int_{\Omega_0} p_\lambda^2 \, dx$$

Using the definition of the norm on the first term and integrating by parts on the second term we have that

$$\frac{1}{2} \frac{d}{dt} \|p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2 = \kappa \left[\int_{\Gamma} p_\lambda (\nabla p_\lambda \cdot \hat{\mathbf{n}}) \, d\sigma - \int_{\Omega} |\nabla p_\lambda|^2 \, dx \right] - \lambda \|p_\lambda(t, \cdot)\|_{L^2(\Omega_0)}^2$$

The boundary conditions on $\partial\Omega$ imply that the first term in the square bracket vanishes. Integrating from 0 to T in t and applying the fundamental theorem of calculus, recalling the definition of the norm (6), we get after rearranging

$$\frac{1}{2}\|p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2 + \kappa\|\nabla p_\lambda\|_{L^2(Q)}^2 + \lambda\|p_\lambda\|_{L^2(Q_0)}^2 = \frac{1}{2}\|p_\lambda(0, \cdot)\|_{L^2(\Omega)}^2 \quad (14)$$

Recall that $p_\lambda(0, x) := E_0[g](x)$ and note that from (14), since all the terms on the left are positive, it follows that

$$\sup_{t \in [0, T]} \|p_\lambda(t)\|_{L^2(\Omega)}^2 \leq K_1 \quad (15)$$

$$\|\nabla p_\lambda\|_{L^2(Q)}^2 \leq K_2 \quad (16)$$

$$\|p_\lambda\|_{L^2(Q_0)}^2 \leq \frac{K_1}{2\lambda} \quad (17)$$

Here $K_1 = \|g(x)\|_{L^2(\Omega)}^2$ and $K_2 = K_1/2\kappa$. The above estimates then show that $p_\lambda(t, x)$ is uniformly bounded in $L^\infty(I; L^2(\Omega))$, $\nabla p_\lambda(t, x)$ is uniformly bounded in $L^2(Q)$ and that $\|p_\lambda\|_{L^2(Q_0)} \rightarrow 0$, $\mathcal{O}(\lambda^{-\frac{1}{2}})$, as $\lambda \rightarrow \infty$ thus proving the theorem. \square

If we impose additional regularity in the initial condition g we can prove:

Lemma 3.3 (More Uniform L^2 Bounds). *Let $p_\lambda(t, x)$ satisfy the conditions of Lemma 3.2. Additionally, suppose that the initial condition $E_0[g](x)$ is in $H^1(\Omega)$. Then for (almost) every $t > 0$ we have that $p_\lambda(t, \cdot)$ and $\nabla p_\lambda(t, \cdot)$ are uniformly bounded in $L^2(\Omega_0)$ and $L^2(\Omega)$ respectively with $\|p_\lambda(t, \cdot)\|_{L^2(\Omega_0)} \rightarrow 0$ as $\lambda \rightarrow \infty$. In addition, $\frac{\partial p_\lambda}{\partial t}$ is uniformly bounded in $L^2(Q)$.*

Remark 1. In light of Corollary 2.2 the condition that $E_0[g] \in H^{1,0}$ is equivalent to the trace of g vanishing on Γ_0 .

Proof. In the same vein as the proof of Lemma 3.2 we multiple equation (10) by $\frac{\partial p_\lambda}{\partial t}$ and integrate over Ω to obtain

$$\int_{\Omega} \left(\frac{\partial p_\lambda}{\partial t} \right)^2 dx = \kappa \int_{\Omega} \frac{\partial p_\lambda}{\partial t} \Delta p_\lambda dx - \lambda \int_{\Omega_0} p_\lambda \frac{\partial p_\lambda}{\partial t} dx$$

Again we perform an integration by parts on the second term (we will from now drop the integration measures)

$$\int_{\Omega} \left(\frac{\partial p_\lambda}{\partial t} \right)^2 = \kappa \left[\int_{\Gamma} \frac{\partial p_\lambda}{\partial t} \nabla p_\lambda \cdot \hat{n} - \sum_{i=1}^3 \int_{\Omega} \frac{\partial p_\lambda}{\partial x_i} \frac{\partial^2 p_\lambda}{\partial x_i \partial t} \right] - \lambda \int_{\Omega_0} p_\lambda \frac{\partial p_\lambda}{\partial t} \quad (18)$$

Again by the boundary conditions the first term in the square bracket vanishes. We deal with the second term by noting that by a theorem of OLENIK [9, 13] we can switch the order of the time and space differentiation so that

$$\sum_{i=1}^3 \int_{\Omega} \frac{\partial p_\lambda}{\partial x_i} \frac{\partial^2 p_\lambda}{\partial x_i \partial t} = \sum_{i=1}^3 \int_{\Omega} \frac{\partial p_\lambda}{\partial x_i} \frac{\partial^2 p_\lambda}{\partial t \partial x_i} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla p_\lambda|^2 := \frac{1}{2} \frac{d}{dt} \|\nabla p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2$$

Hence it follows that (18) becomes

$$\left\| \frac{\partial p_\lambda(t, \cdot)}{\partial t} \right\|_{L^2(\Omega)}^2 = -\frac{\kappa}{2} \frac{d}{dt} \|\nabla p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2 - \frac{\lambda}{2} \frac{d}{dt} \|p_\lambda(t, \cdot)\|_{L^2(\Omega_0)}^2.$$

Integrating from 0 to T and using the FTC again we then obtain

$$\left\| \frac{\partial p_\lambda}{\partial t} \right\|_{L^2(Q)}^2 + \frac{\kappa}{2} \|\nabla p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|p_\lambda(t, \cdot)\|_{L^2(\Omega_0)}^2 = \frac{\kappa}{2} \|\nabla p_\lambda(0, \cdot)\|_{L^2(\Omega)}^2 \quad (19)$$

since $p_\lambda(0, x) = 0$ in Ω_0 . From (19), it follows that

$$\sup_{t \in [0, T]} \|\nabla p_\lambda(t, \cdot)\|_{L^2(\Omega)}^2 \leq C_1 \quad (20)$$

$$\left\| \frac{\partial p_\lambda}{\partial t} \right\|_{L^2(Q)}^2 \leq C_2 \quad (21)$$

$$\sup_{t \in [0, T]} \|p_\lambda(t, \cdot)\|_{L^2(\Omega_0)}^2 \leq \frac{C_3}{\lambda} \quad (22)$$

The above estimates show that $\nabla p_\lambda(t, x)$ is uniformly bounded in $L^\infty(I; L^2(\Omega))$, $\frac{\partial p_\lambda}{\partial t}(t, x)$ is uniformly bounded in $L^2(Q)$ and $p_\lambda(t, x) \rightarrow 0$ in $L^\infty(I; L^2(\Omega_0))$, $\mathcal{O}(\lambda^{-\frac{1}{2}})$ as $\lambda \rightarrow \infty$. \square

Entirely similar computations to the ones in the preceding Lemmas allow us to obtain:

Lemma 3.4. *Under the above conditions on g , the solution to (8) and (9) satisfies $\rho(t, x) \in L^\infty(I; H_o^1(\Omega_1)) \cap H^1(I; L^2(\Omega_1))$. In particular*

$$\frac{1}{2} \sup_{t \in I} \|\rho(t, \cdot)\|_{L^2(\Omega_1)}^2 + \kappa \|\nabla \rho\|_{L^2(Q_1)}^2 = \frac{1}{2} \|g\|_{L^2(\Omega_1)}^2$$

and

$$\left\| \frac{\partial \rho}{\partial t} \right\|_{L^2(Q_1)}^2 + \frac{\kappa}{2} \sup_{t \in I} \|\nabla \rho(t, \cdot)\|_{L^2(\Omega_1)}^2 = \frac{\kappa}{2} \|\nabla g\|_{L^2(\Omega_1)}^2.$$

4 Weak Convergence

We can now derive various results as implications of the preceding Lemmas. We will always implicitly assume that the conditions under which Lemmas 3.2 and 3.3 were derived hold. We begin with:

Theorem 4.1 (Weak Convergence). *There exist p^* and subsequence of $\{p_\lambda\}$ such that $p_\lambda \rightharpoonup p^*$ weakly in $H^{1,1}(Q)$ and $p_\lambda \rightharpoonup p^*$ weak - * in $L^\infty(I; H^1(\Omega))$*

Proof. Lemmas 3.2 and 3.3 imply that p_λ is uniformly bounded in $L^\infty(I; H^1(\Omega))$. Thus weak - * convergence follows: that is for each $\phi \in L^1(I; H^1(\Omega))$ and, passing, if necessary, to a subsequence, we have that

$$\int_0^T \int_\Omega [p_\lambda \phi + \nabla p_\lambda \cdot \nabla \phi] dx dt \longrightarrow \int_0^T \int_\Omega [p^* \phi + \nabla p^* \cdot \nabla \phi] dx dt$$

Now as T is finite, we have the elementary embedding,

$$L^\infty(I; L^2(\Omega)) \hookrightarrow L^2(Q), \quad (23)$$

since

$$\|u\|_{L^2(Q)}^2 := \int_0^T \int_\Omega |u(t, x)|^2 dx dt \leq T \|u\|_{L^\infty(I; L^2(\Omega))}^2.$$

This implies that p_λ is also a uniformly bounded sequence in $L^2(I; H^1(\Omega))$. We also have that p_λ is a uniformly bounded sequence in $H^1(I; L^2(\Omega))$ and thus uniformly bounded in $H^1(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))$ as well. By weak compactness, since bounded sequences in a reflexive Banach space have a weakly convergent subsequence, the existence of a weak limit then follows. \square

Remark 2. Of course, by the Sobolev embedding theorem, it follows that p_λ is also a uniformly bounded sequence in $C^0([0, T]; L^2(\Omega))$ and we will always have this in mind.

We shall presently show that in the binding region Q_0 and on the boundary Σ_0 we can get convergence in more regular Sobolev spaces at different rates which turn out to depend on the regularity demanded by the space under consideration.

Lemma 4.1. *Let $0 < \epsilon_1, \epsilon_2 < 1$ be fixed. Then $\|p_\lambda^+\|_{H^{\epsilon_1, \epsilon_2}(Q_0)} \rightarrow 0$, $\mathcal{O}(\lambda^{-\epsilon_0})$. If in addition $\frac{1}{2} < \epsilon_1 < 1$, then $\|\mathcal{T}_0^x p_\lambda^+\|_{H^{\sigma_1, \sigma_2}(\Sigma_0)} \rightarrow 0$, $\mathcal{O}(\lambda^{-\epsilon_0})$, where $\epsilon_0 := \min(\frac{1-\epsilon_1}{2}, \frac{1-\epsilon_2}{2})$, $\sigma_1 := \epsilon_1 - \frac{1}{2}$ and $\sigma_2 := \epsilon_2(1 - \frac{1}{2\epsilon_1})$.*

Proof. This basically follows from the interpolation lemmas. First note that since p_λ^+ is uniformly bounded in $H^{1,1}(Q_0)$ we have by Lemma 2.2 that

$$[L^2(Q_0), H^1(I; L^2(\Omega_0))]_{\epsilon_2, 2} = H^{\epsilon_2}(I; L^2(\Omega_0)),$$

and by Lemma 2.1 that

$$\begin{aligned} \|p_\lambda^+\|_{H^{\epsilon_2}(I; L^2(\Omega_0))} &\leq C \|p_\lambda^+\|_{L^2(Q_0)}^{1-\epsilon_2} \|p_\lambda^+\|_{H^1(I; L^2(\Omega_0))}^{\epsilon_2} \\ &\leq C \lambda^{-\frac{(1-\epsilon_2)}{2}}. \end{aligned}$$

Similarly we have, again by interpolation, that

$$[L^2(Q_0), L^2(I; H^1(\Omega_0))]_{\epsilon_1, 2} = L^2(I; H^{\epsilon_1}(\Omega_0)),$$

and

$$\begin{aligned} \|p_\lambda^+\|_{L^2(I; H^{\epsilon_1}(\Omega_0))} &\leq C \|p_\lambda^+\|_{L^2(Q_0)}^{1-\epsilon_1} \|p_\lambda^+\|_{L^2(I; H^1(\Omega_0))}^{\epsilon_1} \\ &\leq C \lambda^{-\frac{(1-\epsilon_1)}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} \|p_\lambda^+\|_{H^{\epsilon_1, \epsilon_2}(Q_0)} &= \left(\|p_\lambda^+\|_{L^2(I; H^{\epsilon_1}(\Omega_0))}^2 + \|p_\lambda^+\|_{H^{\epsilon_2}(I; L^2(\Omega_0))}^2 \right)^{\frac{1}{2}} \\ &\leq \left(C \lambda^{-(1-\epsilon_1)} + C \lambda^{-(1-\epsilon_2)} \right)^{\frac{1}{2}} = \mathcal{O}(\lambda^{-\epsilon_0}), \end{aligned}$$

where ϵ_0 is as defined in the statement of the lemma. With these estimates a direct application of the trace theorem, Theorem 2.1, now yields that for $\epsilon_1 > \frac{1}{2}$

$$\|\mathcal{T}_0^x p_\lambda^+\|_{H^{\sigma_1, \sigma_2}(\Sigma_0)} \leq C \|p_\lambda^+\|_{H^{\epsilon_1, \epsilon_2}(Q_0)} = \mathcal{O}(\lambda^{-\epsilon_0}), \quad (24)$$

where $\sigma_1 = \epsilon_1 - \frac{1}{2}$ and $\sigma_2 = \epsilon_2(1 - \frac{1}{2\epsilon_1})$. \square

We have so far shown the existence of a weak limit p^* . We would very much like for this limit to coincide with ρ . This is the content of the next lemma:

Lemma 4.2. $p_\lambda \rightharpoonup E_0[\rho]$ weakly.

Proof. Lemma 4.1 implies that for any $0 \leq \sigma_1, \sigma_2 < \frac{1}{2}$, $\|\mathcal{T}_0^x p_\lambda^+\|_{H^{\sigma_1, \sigma_2}(\Sigma_0)} \rightarrow 0$ or in other words that $\|\mathcal{T}_0^x p^*\|_{H^{\sigma_1, \sigma_2}(\Sigma_0)} = 0$. Since p_λ^+ is uniformly bounded in $H^{1,1}(Q_0)$, we have that $p_\lambda|_{\Sigma_0}$ is uniformly bounded in $H^{\frac{1}{2}, \frac{1}{2}}(\Sigma_0)$. Since $H^{\frac{1}{2}, \frac{1}{2}}(\Sigma_0) \hookrightarrow H^{\sigma_1, \sigma_2}(\Sigma_0)$ embeds compactly for $0 \leq \sigma_1, \sigma_2 < \frac{1}{2}$, it follows that $p^*|_{\Sigma_0} = 0$ in $H^{\frac{1}{2}, \frac{1}{2}}(\Sigma_0)$ and thus $p^* \in H_o^{1,0}(Q_1)$.

Now for any $\psi \in H^1(Q_1)$ with $\psi|_{\Sigma_0} = 0$ and $\psi|_{\Omega_1, \{T\}} = 0$, let $\phi = E_0[\psi]$ and note that $\phi \in H^1(Q)$ in view of Corollary 2.2 and that $\phi|_{\Omega_0} = 0$. Using the definition of a weak solution and the fact we have weak convergence we get:

$$\begin{aligned} 0 &= \int_Q (\kappa \nabla p_\lambda \cdot \nabla \phi - p_\lambda \partial_t \phi) - \int_{\Omega_{\{0\}}} E_0[g] \phi + \lambda \int_{Q_0} p_\lambda \phi, \\ &= \lim_{\lambda \rightarrow \infty} \left[\int_Q (\kappa \nabla p_\lambda \cdot \nabla \phi - p_\lambda \partial_t \phi) - \int_{\Omega_{\{0\}}} E_0[g] \phi \right], \\ &= \int_{Q_1} (\kappa \nabla p^* \cdot \nabla \psi - p^* \partial_t \psi) - \int_{\Omega_{1, \{0\}}} g \psi. \end{aligned}$$

Hence p^* is a weak solution to (8) and (9) and by uniqueness $p^* = \rho$. \square

5 Strong Convergence

5.1 An Optimal Rate

So far we have shown that $p_\lambda \rightarrow \rho$ weakly and $p_\lambda|_{Q_0} \rightarrow 0$ strongly. We would like to transfer this strong convergence on the interior, i.e. Q_0 , to the exterior domain, Q_1 .

Define the error, e between the two solutions outside the binding region: $e(t, x) := p_\lambda^-(t, x) - \rho(t, x)$ for $x \in \Omega_1$. It follows from equations (8) and (12) that e satisfies:

$$\frac{\partial e}{\partial t} = \kappa \Delta e, \quad (t, x) \in Q_1, \quad (25)$$

with the boundary conditions:

$$\left. \begin{aligned} e(t, x) &= p_\lambda^+(t, x), & (t, x) &\in \Sigma_0, \\ \nabla e(t, x) \cdot \hat{\mathbf{n}} &= 0, & (t, x) &\in \Sigma, \end{aligned} \right\} \quad (26)$$

and with the initial condition $e(0, x) \equiv 0$. Thus e satisfies a homogenous heat equation in Q_1 and is basically controlled by the behavior of p_λ^+ on Σ_0 . Thus if $p_\lambda^+|_{\Sigma_0} \rightarrow 0$ we should get that $e \rightarrow 0$. The classical theory for such equations requires that the boundary data be in $H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$. Unfortunately, Lemma 4.1, in particular equation (24), only shows that $\|p_\lambda^+|_{\Sigma_0}\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)} = \mathcal{O}(1)$. Our task is to extend the solvability theory for the above equations in order to include weaker boundary spaces were we happen to have stronger convergence rates.

We begin by noting that

Lemma 5.1 (L^2 boundary convergence).

$$\|\mathcal{T}_0^x[p_\lambda^+]\|_{L^2(\Sigma_0)} = \mathcal{O}(\lambda^{-\frac{1}{4}}), \text{ as } \lambda \rightarrow \infty.$$

Proof. By Theorem 2.2 we have that

$$\|\mathcal{T}_0^x[p_\lambda]\|_{L^2(\Sigma_0)} \leq C_\epsilon \|p_\lambda\|_{L^2(Q_0)} + \epsilon \|\nabla p_\lambda\|_{L^2(Q_0)} \quad (27)$$

By Lemmas 3.2 and 3.3 we have that $\|\nabla p_\lambda\|_{L^2(Q_0)}$ is uniformly bounded and that

$$\|p_\lambda\|_{L^2(Q_0)} \leq \frac{C}{\sqrt{\lambda}},$$

which tends to 0 as $\lambda \rightarrow \infty$. Thus it follows from (27) that for *suitable* choice of $\epsilon > 0$,

$$\overline{\lim} \|\mathcal{T}_0^x[p_\lambda]\|_{L^2(\Sigma_0)} \leq \tilde{C}\epsilon, \quad (28)$$

and this gives

$$\lim_{\lambda \rightarrow \infty} \|\mathcal{T}_0^x[p_\lambda]\|_{L^2(\Sigma_0)} = 0.$$

Given that we want (28) to hold, we require from (27) that

$$\overline{\lim} \frac{2C_\epsilon K}{\sqrt{\lambda}} = 0.$$

However, $C_\epsilon = 1/\epsilon$ and thus the above equation is satisfied only if ϵ is $\mathcal{O}(\lambda^{\delta - \frac{1}{2}})$ for any $0 < \delta < 1/2$. The optimal choice is $\delta = 1/4$ from which we get that $\|\mathcal{T}_0^x[p_\lambda]\|_{L^2(\Sigma_0)} = \mathcal{O}(\lambda^{-1/4})$ as desired. \square

Remark 3. We needed the previous Lemma out of a slight but important technicality. Formally, in (24) we could have taken $\epsilon_1 = \frac{1}{2}$ to get precisely the same estimate in the previous Lemma. However, the trace map ceases to be a bounded operator when $\epsilon_1 = \frac{1}{2}$, hence the roundabout argument. The convergence rate obtained is obviously optimal within our framework.

5.2 Very Weak Solutions

Consider the following abstract parabolic problem:

$$\left. \begin{aligned} (\partial_t - \Delta)u &= f & (t, x) \in Q_1 \\ u &= h & (t, x) \in \Sigma_0 \\ \nabla u \cdot \hat{n} &= 0 & (t, x) \in \Sigma \\ u(0, x) &\equiv u_0(x) \end{aligned} \right\} \quad (29)$$

We will now give a way to define solutions with rather “rough” inhomogenous data $f(t, x)$, $h(t, x)$ and $u_0(x)$. The technique is essentially a time reversal and transposition argument and is due to [10] but we follow the development of FRENCH & KING [8] and BERGGREN [2]. Let $v \in L^2(Q_1)$ be arbitrary. Consider, first, the solution operator

$$\begin{aligned} \mathcal{S} : L^2(Q_1) &\rightarrow H_o^{2,1}(Q_1) \\ v &\mapsto w \end{aligned}$$

where $w(t, x)$ the unique solution of the auxiliary problem:

$$\left. \begin{aligned} (-\partial_t - \Delta)w &= v & (t, x) \in Q_1 \\ w &= 0 & (t, x) \in \Sigma_0 \\ \nabla w \cdot \hat{n} &= 0 & (t, x) \in \Sigma \\ w(T, x) &= 0. \end{aligned} \right\} \quad (30)$$

We assume that the map \mathcal{S} exists and is bounded. Using this solution operator we now define the following map:

$$\begin{aligned} \mathcal{L} : L^2(Q_1) &\rightarrow H_o^1(\Omega_1) \times H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0) \\ v &\mapsto \left((\mathcal{S}v)|_{\Omega_1, \{0\}}, \frac{\partial(\mathcal{S}v)}{\partial \mathbf{n}} \Big|_{\Sigma_0} \right) \end{aligned} \quad (31)$$

It follows from Theorem 2.1 that \mathcal{L} is a bounded linear map since it is a composition of bounded linear maps.

Now suppose that the data $f(t, x)$, $h(t, x)$ and $u_0(x)$ of (29) are in $L^2(Q_1)$, $H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)$ and $(H_o^1(\Omega_1))^*$ respectively the dual spaces of $L^2(Q_1)$, $H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)$ and $H_o^1(\Omega_1)$. We can then define the linear functional:

$$B_{f,g,u_0}(v)^1 := \int_{\Omega_1} u_0(x)w(x, 0) + \int_{Q_1} f(t, x)w(t, x) + \int_{\Sigma_0} h(t, x) \frac{\partial w}{\partial \mathbf{n}}(t, x) \quad (32)$$

where $w = \mathcal{S}(v)$.

Definition 5.1. We say that u is a *very weak* solution of (29) if

$$B_{f,g,u_0}(v) = \int_{Q^-} u(t, x)v(t, x) \quad \forall v \in L^2(Q^-) \quad (33)$$

¹Strictly speaking these integrals should be parings between the spaces and their duals

As motivation for our definition of *very weak* solutions, let w and v satisfy (30). For any ϕ smooth enough satisfying $\nabla \phi \cdot \mathbf{n}|_{\Sigma} = 0$ an integration by parts gives

$$\begin{aligned}
\int_{Q_1} \phi v &\equiv \int_{Q_1} (-\partial_t - \Delta) w \phi \\
&= - \int_{\Omega_{1,\{T\}}} \phi w + \int_{\Omega_{1,\{0\}}} \phi w + \int_0^T \int_{\Omega_1} w \partial_t \phi - \int_0^T \left[\int_{\partial\Omega_1} \phi \nabla w \cdot \mathbf{n} - \int_{\Omega_1} \nabla w \cdot \nabla \phi \right] \\
&= \int_{\Omega_{1,\{0\}}} \phi w + \int_{Q_1} \partial_t \phi w + \int_{\Sigma_0} \phi \frac{\partial w}{\partial \mathbf{n}} + \int_0^T \left[\int_{\partial\Omega_1} w \nabla \phi \cdot \mathbf{n} - \int_{\Omega_1} w \Delta \phi \right] \\
&= \int_{\Omega_{1,\{0\}}} \phi w + \int_{Q_1} (\partial_t - \Delta) \phi w + \int_{\Sigma_0} \phi \frac{\partial w}{\partial \mathbf{n}}
\end{aligned}$$

Remark 4. This definition for a weak solution should be contrasted with the one previously given. First of all, the regularity requirement of the solution is weaker. Secondly, the “test functions” and the solution lie in the same space.

We now have the following existence theorem:

Theorem 5.1. *The equations (29) admit a unique very weak solution $u \in L^2(Q_1)$.*

Proof. First we note that B_{f,g,u_0} is a bounded linear functional on $L^2(Q_1)$ since

$$\begin{aligned}
|B_{f,g,u_0}(v)| &= \left| \int_{\Omega_1} u_0(x) w(0, x) + \int_{Q_1} f(t, x) w(t, x) + \int_{\Sigma_0} h(t, x) \frac{\partial w}{\partial \mathbf{n}}(t, x) \right| \\
&\leq \left| \int_{\Omega_1} u_0(x) w(0, x) \right| + \left| \int_{Q_1} f(t, x) w(t, x) \right| + \left| \int_{\Sigma_0} h(t, x) \frac{\partial w}{\partial \mathbf{n}}(t, x) \right| \\
&\leq \|u_0\|_{(H_o^1(\Omega_1))^*} \|w(0, \cdot)\|_{H_o^1(\Omega_1)} + \|f\|_{L^2(Q_1)} \|w\|_{L^2(Q_1)} \\
&\quad + \|h\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)} \left\| \frac{\partial w}{\partial \mathbf{n}} \right\|_{H^{\frac{1}{2}, \frac{1}{4}}(\Sigma_0)} \\
&\leq C_1 \|u_0\|_{(H_o^1(\Omega_1))^*} \|w\|_{H^{2,1}(Q_1)} + \|f\|_{L^2(Q_1)} \|w\|_{H^{2,1}(Q_1)} \\
&\quad + C_2 \|h\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)} \|w\|_{H^{2,1}(Q_1)} \\
&\leq C \left(\|u_0\|_{(H_o^1(\Omega_1))^*} + \|f\|_{L^2(Q_1)} + \|h\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)} \right) \|w\|_{H^{2,1}(Q_1)} \\
&\leq C \left(\|u_0\|_{(H_o^1(\Omega_1))^*} + \|f\|_{L^2(Q_1)} + \|h\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)} \right) \|v\|_{L^2(Q_1)}
\end{aligned}$$

It then follows by the Riesz representation theorem that there exists a unique $u \in L^2(Q_1)$ such that

$$B_{f,g,u_0}(v) = (u, v)_{L^2(Q_1)} := \int_{Q_1} u(t, x) v(t, x) \quad \forall v \in L^2(Q_1)$$

and we have the estimate that

$$\|u\|_{L^2(Q_1)} \leq C \left(\|u_0\|_{(H_o^1(\Omega_1))^*} + \|f\|_{L^2(Q_1)} + \|h\|_{H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma_0)} \right) \quad (34)$$

□

Thus, we immediately obtain the following which can be regarded as the main results of this paper:

Theorem 5.2 (Main result I). *The difference, e , between the two solutions satisfies:*

$$\|e\|_{L^2(Q_1)} := \|p_\lambda^- - \rho\|_{L^2(Q_1)} \leq C \|\mathcal{T}_0^x[p_\lambda^+]\|_{L^2(\Sigma_0)} = \mathcal{O}(\lambda^{-\frac{1}{4}}) \quad (35)$$

Proof. Simply apply Theorem 5.1 with $f = 0$, $u_0 = 0$ and $h = p_\lambda^+|_{\Sigma_0}$ and use (34). □

Theorem 5.3 (Main Result II). *Fix $0 < \delta_1, \delta_2 < 1$. Then $\|p_\lambda^- - \rho\|_{H^{\delta_1, \delta_2}(Q_1)} \rightarrow 0$, $\mathcal{O}(\lambda^{-\delta_0})$ where $\delta_0 := \min(\frac{1-\delta_1}{4}, \frac{1-\delta_2}{4})$.*

Proof. Since $p_\lambda - E_0[\rho] \rightharpoonup 0$ weakly in $H^{1,1}$ it follows $p_\lambda^- - \rho$ is a uniformly bounded sequence and that, for some C (which can be obtained from Lemmas 3.2 and 3.3), we have $\|p_\lambda^- - \rho\|_{H^{1,1}(Q_1)} \leq C$. Theorem 5.2 now gives $\|p_\lambda^- - \rho\|_{L^2(Q_1)} = \mathcal{O}(\lambda^{-\frac{1}{4}})$ and the result follows by interpolation as in the proof of Lemma 4.1. □

6 Final Remarks

We have not dwelt on regularity issues in this paper. Although such issues are interesting and important, we feel that they carry us too far afield from our goal which is to obtain estimates for convergence rates. It seems to be the case that one would require certain more technical tools in order to discuss regularity problems and we postpone such a treatment to a future paper.

Another salient question is the optimality of the rates we have obtained here. In the paper by DEMUTH, KIRSCH & MCGILLIVRAY mentioned in the introduction the rate obtained is $\mathcal{O}(\lambda^{-\frac{1}{2}+\sigma})$ (in our notation), where $0 < \sigma < \frac{1}{2}$ is a constant depending on certain geometric conditions on Γ_0 . We note here that they consider the semigroup difference (in our notation): $\|R_1 \circ e^{-\kappa t A_\lambda} - e^{-\kappa t B} \circ R_1\|$ as an operator acting *now in* $L^2(\Omega)$. They also show that if Γ_0 is smooth that one essentially has $\sigma = 0$ and that if Γ_0 is (uniformly) convex then $\sigma = 1/4$ is optimal (see also [6]). It will be interesting to extend their results using a more “P.D.E approach”.

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